OPPORTUNITY TO LEARN FUNCTION TRANSFORMATIONS IN DYNAMIC MATHEMATICAL ENVIRONMENT: AN ANALYSIS OF 347 GEOGEBRA APPLETS

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This study examined the opportunity to learn function transformations afforded by the GeoGebra applets available on the GeoGebra website. Our analysis focused on the functions and their representations through which function transformations are explored in these GeoGebra applets, the effects and components of transformations that the GeoGebra applets afford students to learn, and the scaffolding provided in these applets. The results show that function transformations in the GeoGebra applets are often explored in the context of function families (e.g., quadratic and trigonometric functions) that use specific representations (e.g., graphical and symbolic). Moreover, the defining parameters of transformations and corresponding points on the graphs of parent and child functions are not visible in most GeoGebra applets. Only a small number of GeoGebra applets include questions or tasks that aim to scaffold students' exploration of function transformations. These results invite us to rethink how to design GeoGebra applets that maximize students' opportunity to learn function transformation conceptually in dynamic mathematical environments.

Keywords: Opportunity to Learn; Function Transformations; Technology

Function transformation is an important mathematical concept for secondary and post-secondary students to learn. Function transformation together with function concept, covariation and rate of change, families of functions, and multiple representations of functions are the big ideas that are essential for students to understand functions (Lloyd, Beckmann, Zbiek, &Cooney, 2010). The study of function transformation can provide students with opportunities to use, reflect on, and possibly modify their understanding of function. Moreover, the study of function transformation can provide students with opportunities to build connections between transformations of a geometric shape and transformations of the graph of a function so that students can see how a single mathematical idea manifests in various domains of mathematics. Such experience can support students to perceive mathematics as a field of intricately related ideas rather than a collection of compartmentalized concepts and procedures. Given its importance, many countries (e.g., the United States, Australia, Turkey, and China) emphasize function transformations in their curriculum standards. For instance, in the United States, the Common Core State Standards for Mathematics (NGA Center and CCSSO, 2010) requires students to "identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $k \cdot f(x)$, $f(k \cdot x)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs; experiment with cases and illustrate an explanation of the effects on the graph using technology" (p.70). However, researchers have found that students experience difficulties in understanding function transformations, including but not limited to recognizing which transformation is applied to a parent function, identifying properties of function under transformation, and providing a mathematically sound explanation for the rules of function transformations (Lage & Trigueros, 2006; Zazkis, Liljedahl, & Gadowsky, 2003). These difficulties come from different sources, including an insufficient understanding of the function concept (Boz-Yaman & Yigit Koyunkaya, 2019; Kimani 2008; Lage & Gaisman, 2006; Baker et al., 2001), problematic conceptions of symbols involved in

transformations (Yim &Lee, 2021), and inappropriate instructional approach (Hall & Giacin, 2013; Borba & Confrey, 1996).

The use of mathematics action technologies (Dick & Hollebrands, 2011) such as graphing calculators, computer algebra systems, and dynamic mathematical environments (e.g., GeoGebra and the Geometer's Sketchpad) has the potential to transform how mathematics is taught and learned. Meanwhile, researchers have pointed out that the realization of such potentiality heavily depends on the design of appropriate teaching/learning tasks and resources with these technologies (Günster & Weigand, 2020; Leung & Bolite-Frant, 2015). The use of different tasks and recourses might impact not only what mathematical ideas are learned but also how they are learned. Therefore, it is important to analyze the opportunity to learn afforded by tasks and resources that are designed with the use of a mathematics action technology. As an interactive geometry, algebra, statistics, and calculus application that can run on multiple platforms (e.g., desktops, tablets, and online), GeoGebra is developed for learning and teaching mathematics from primary school to university level. Although GeoGebra users can create activities from scratch, existing GeoGebra applets are important resources for educators because GeoGebra cloud service allows users to upload and share GeoGebra applets with others. Currently, it hosts more than one million free activities, simulations, exercises, lessons, and games for mathematics and science. A significant number of these GeoGebra applets are created for students to learn function transformations. Given that GeoGebra is a community of millions of students and teachers who are potential users of these applets, it is important to examine the opportunity to learn function transformations afforded by publicly available GeoGebra applets from the GeoGebra website.

Theoretical Background and Research Questions

Opportunity-to-Learn Required for Understanding Function Transformations

When particular learning goals are not achieved by students, it is plausible to ask whether they have received the learning experience that enables them to develop the competencies expressed in these goals. Therefore, it seems natural for researchers to introduce the notion of opportunity-to-learn (OTL). Indeed, this concept was coined by Carroll (1963) when referring to sufficient time for students to learn (Walkowiak, Pinter, & Berry, 2017). Since then, the notion of OTL has been interpreted from multiple theoretical perspectives, where the focus may be on cognitive, curriculum and assessment design, social or affective dimensions of learning, issues of equity and access, or the broad policy and political contexts of learning and teaching (Goos, 2014). At the core of the OTL concept is the need for high-quality curriculum and instructional resources and teaching to enable students to achieve rigorous learning goals. Brewer and Stasz (1996) considered three OTL variables: (1) curriculum content coverage, that is, whether students have been taught the subjects and topics essential to attain the standards; (2) instructional strategies, that is, whether students have experience with particular kinds of tasks and solution processes; (3) the quality of instructional resources. This study focused on the quality of a particular type of instructional resource, namely, online GeoGebra applets. The existing GeoGebra resources were chosen because of the large number of GeoGebra users and the high volume of publicly available GeoGebra resources on function transformations. To our knowledge, these existing GeoGebra resources have not been carefully studied. This study aimed to examine the quality of GeoGebra resources on function transformations to understand how they afford or constrain students' OTL about function transformations.

Function transformations are mathematical operations (i.e., translation, reflection, rotation, and dilation) that can be performed on an existing function and its graph to give a modified version of that function and its graph that has a similar shape to the original function. To understand function

transformations, students need to have the opportunity to understand both the function and the transformation performed on it. Since different function families are included in the secondary mathematics curriculum, it is important to consider what functions are used to explore function transformations. Moreover, like other mathematical objects, the function concept cannot be directly perceived with instruments and access to the function concept relies on the use of a system of semiotic representations (Duval, 2017). Therefore, representations of parent and child functions impact students' opportunity to learn function transformations. Functions can be represented by words, tables, symbols, or graphs, each of which has its advantages. Graphs provide a visual representation of a function, showing how the function changes over a range of inputs. Symbolic representation uses an equation to compactly express a function relation that makes it easy to compute functional values. By using two columns (one with the dependent variable and the other with the independent variable), tables explicitly supply the functional values of specific inputs. Building connections among multiple representations of a parent function and its child function is critical for understanding function transformations. Besides the opportunity to learn parent and child functions and their representations, students also need to have the opportunity to learn transformations performed on functions. This includes the opportunity to learn not only the effects of different transformations but also what defines each transformation. For instance, understanding function translation requires the opportunity to learn not only the effect of translation as a motion but also what defines a translation (i.e., translation vector). Similarly, understanding function dilation requires the opportunity to learn not only the effect of dilation as a motion but also what defines a dilation (i.e., a center of dilation and a scale factor). When examining the OTL afforded or constrained by instructional resources, it is important to analyze whether and in what ways the resources explicitly provide students with the opportunity to learn different aspects of function transformations.

Approaches to Function Transformations

Within curriculum and instructional resources, different approaches have been taken to function transformation. These approaches afford different opportunities to learn function transformation. The transformation of functions can be taught through a graphical approach (Heid, Wilson, & Blume, 2015). In this approach, graphing utility is used to examine the graphs of a parent function and its child function simultaneously. Simultaneous display of the graphs of the two functions for different values of the parameters of a child function illustrates the relationship between the graph of the child function and the graph of the parent function for particular parameter values. Relying on visual and numerical clues, the students are assumed to connect the movement of the graph with the change in the numerical values. A tabular approach might also be used to examine function transformation (Zazkis et al., 2003; Heid et al., 2015), in which a table of values is created by plugging numbers into the equations, and then points are plotted on the coordinate plane. Take the horizontal translation of a quadratic function $f(x) = x^2$ 3 units right as an example. The table can show that $x - 3$ will produce the same output values as x, just 3 units "earlier." Thus, f(x – 3) will have the same output values as f(x), but 2 units "later". The points formed by the ordered pairs of input and output values demonstrate that the graph of $(x - 3)^2$ is 3 units to the right of the graph of x^2 as a result of each value of x – 3 being 3 less than the corresponding value of x. It is worth noting that both the graphical and the tabular approaches make use of visual and numerical patterns to reveal the rules for function transformations but do not explain why the patterns exist. Although students might see the patterns, they might still perceive the rules as "counterintuitive" and inconsistent. Nonetheless, both approaches were

frequently used by teachers and students to explain the rules of function transformations (Zazkis et al., 2003).

Zazkis and colleagues (2003) attributed students' difficulties in understanding the horizontal translation of function to the inadequate instructional sequence in which the treatment of transformations of functions is presented in the context of exploring functions rather than in the context of exploring transformations. They proposed a pedagogical approach to function translation that takes transformations as the starting point rather than focuses on the algebraic representation of functions as in the graphical and tabular approaches. According to their approach, students first learn the effect of translation on a set of points in a coordinate plane and are introduced to the formal notation of translation $T(x, y) = (x + a, y + b)$, where a and b are horizontal and vertical components of the motion, respectively. Once the formal notation is introduced, students should be provided ample opportunities to connect the visual image of translation to the mapping rule, which can be achieved by asking students to carry out transformations according to given mappings and to identify mappings according to given visual images. The teacher can then shift students' effort from the translation of a set of points in a geometric shape to the translation of a set of points on the graph of a function. Building on their prior understanding, students will understand that if $P\ (x, y)$ is a point on the graph of a function $y = f(x)$ and $P'\ (x', y')$ is the corresponding point of P under a translation (a, b), then $x' = x + a$ and $y' = y$ +a. Therefore, $x = x' - a$ and $y = y' - b$. Since the point (x, y) is on the graph of $y = f(x)$, the point $(x' - a, y' - b)$ is also on the graph of $y = f(x)$. Thus, we have $y' - b = f(x' - a)$ as the algebraic expression of the function after translation. This approach explains why $f(x - a)$ means shifting the graph of $f(x)$ a unit to the right if a is positive, which appears to be "counterintuitive" if students only focus on the algebraic representation of functions. It is worth noting that this transformation approach can be used to explain the rules for other transformations of function.

The transformation of a function can also be thought of as the transformation of the underlying coordinate axes rather than the graph of the function (Heid et al., 2015). Take the translation of a function as an example. Let $y = f(x)$ be a function graphed in the XY coordinate plane and $P\ (x, y)$ be a point on that graph. The $X'Y'$ coordinate plane is obtained by translating the original coordinate plane by a vector (a, b) for which the origin of the new coordinate plane has coordinates (a, b) in relation to the XY coordinate plane. The coordinates of P in relation to the $X'Y'$ coordinate plane is $(x - a, y - b)$. Thus, the graph of the function f is of the form $y - b = f(x - a)$ in relation to the $X'Y'$ coordinate plane, which can be written as $y = f(x - a) + b$. Other transformations of function can be thought of in a similar way. However, the approach of transforming axes was not frequently used by teachers (Zazkis et al., 2003). Both transformational approaches can avoid the "counterintuitiveness" and "inconsistency" that students frequently experience when dealing with function transformations.

Guided by the notion of OTL and the literature on function transformations, this study aimed to answer the following questions:

1. What are the opportunities to learn function families and their representations afforded by the GeoGebra applets on function transformations?
2. What are the opportunities to learn the effects and components of transformations afforded by the GeoGebra applets on function transformations?
3. To what extent is scaffolding used in the GeoGebra applets to support the opportunity to learn function transformations?

Methodology

Data Collection

GeoGebra applets on function transformations were collected via the built-in search engine provided by GeoGebra in the classroom resources section of their website. The following keywords were used to identify GeoGebra applets on function transformations: *Function Transformation, Function Translation, Function Shift, Function Stretch, Function Shrink, Function Dilation, and Function Rotation.* First, all files were observed and sorted for direct duplication. If a file was a direct duplicate of another file, it was omitted from the data so that every file would be represented once. Direct duplicates were identified in one of two ways. The first way was that the file said "copy of" in the name and the original file was identified apart from the copy. The second way a file was identified as a direct duplicate if the title was changed from the default "copy of" yet the author went unchanged, and the rest of the file was identical to another file the author made. Occasionally, these files also had additional authors, but since there were no changes to the file created by the first author they were also omitted. Second, all files unrelated to function transformations (e.g., transformations of geometric shapes) were also omitted. This process resulted in 347 unique GeoGebra applets on function transformations.

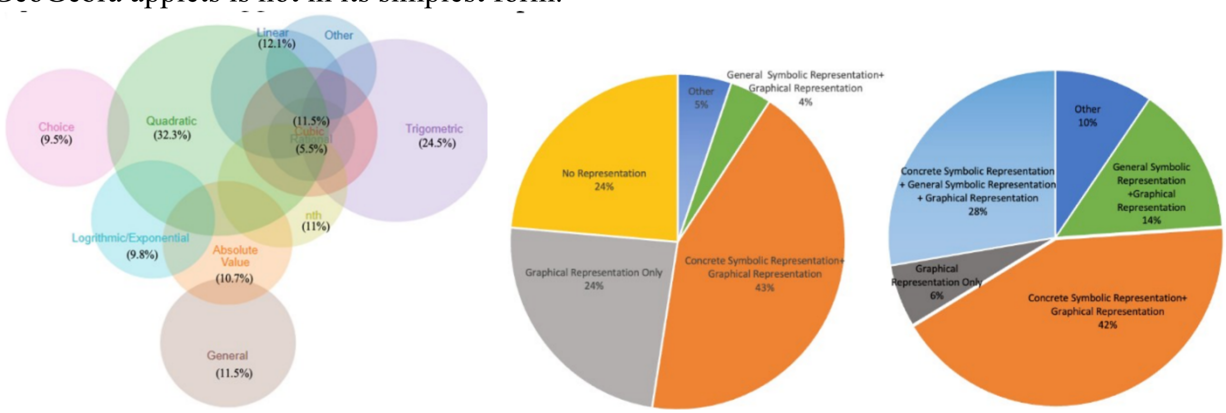
Data Analysis

Building on the literature on function transformation, a coding framework was developed to analyze the opportunity to learn function transformations afforded by each GeoGebra applet. The framework considers the types of functions in which function transformations are explored, the types of representations of the parent and child functions, the effects of transformational motions, the visibility of corresponding points, the visibility of defining parameter(s) of transformations, and the types of scaffolding. The types of functions include specific function families (e.g., linear, quadratic, cubic, absolute value, n^{th} root, rational, exponential, logarithmic, trigonometric, parametric, and step-wise) that are given in the applet, a function that is represented with a general form of function notation (e.g., $g(x) = a \cdot f(b \cdot (x - h)) + k$), and function that can be entered by its user. The types of representations can be graphical, concrete symbolic (e.g., $f(x) = x^2$) general symbolic (e.g., $f(x + h)$), and tabular. It is also possible that no representation of a parent function is given. The effects of transformational motions include horizontal and vertical translation, horizontal and vertical dilation, reflection, and rotation. The coding framework also considers whether these effects of transformations are explicitly noted in a GeoGebra applet. Visibility of corresponding points on the parent and child functions considers whether corresponding points on the parent and child functions are shown in a GeoGebra applet. Visibility of defining parameter(s) of transformations captures whether the defining parameter(s) (e.g., translation vector, line of reflection, and center and scale of dilation) of a transformation are shown in a GeoGebra applet. The types of scaffolding include scaffolding for understanding function transformations, scaffolding for how to interact with a GeoGebra applet, and no scaffolding. It is important to note that many GeoGebra applets in our data set use different function families and multiple representations to explore different types of function transformations. When analyzing these GeoGebra applets, we coded all the function families, representations, and types of function transformations that are afforded by them. Using the coding framework, two researchers first coded the 347 GeoGebra applets independently. Disagreement only occurred in 18 GeoGebra applets and was resolved through discussion.

Results

Opportunity to Learn Functions and their Representations

As shown in Figure 1a, most of the GeoGebra applets include more than one function family by creating check boxes that show/hide different functions. More importantly, particular function families appear in more GeoGebra applets than others. Specifically, 32.3% of the GeoGebra applets use quadratic functions to explore function transformations followed by trigonometric functions (24.5%), linear functions (12.1%), cubic functions (11.5%), general functions (11.5%; e.g., $g(x) = a \cdot f(b \cdot (x - h)) + k$), n^{th} root functions (11.0%), absolute value functions (10.7%), exponential and logarithm functions (9.8%), and rational function (5.5%). 9.5% of the GeoGebra applets also allow their users to enter functions of their own choice. Other functions (i.e., stepwise, parametric, and higher degree polynomial functions) are only included in a very small number of GeoGebra applets (1%~3%). This indicates that more existing GeoGebra applets choose to explore function transformations in the context of quadratic and trigonometric functions. It is worth noting that the parent functions in most GeoGebra applets are in their simplest form (e.g., $f(x) = x^2$, $f(x) = \sin x$, and $f(x) = |x|$). Rarely, a parent function provided by the GeoGebra applets is not in its simplest form.



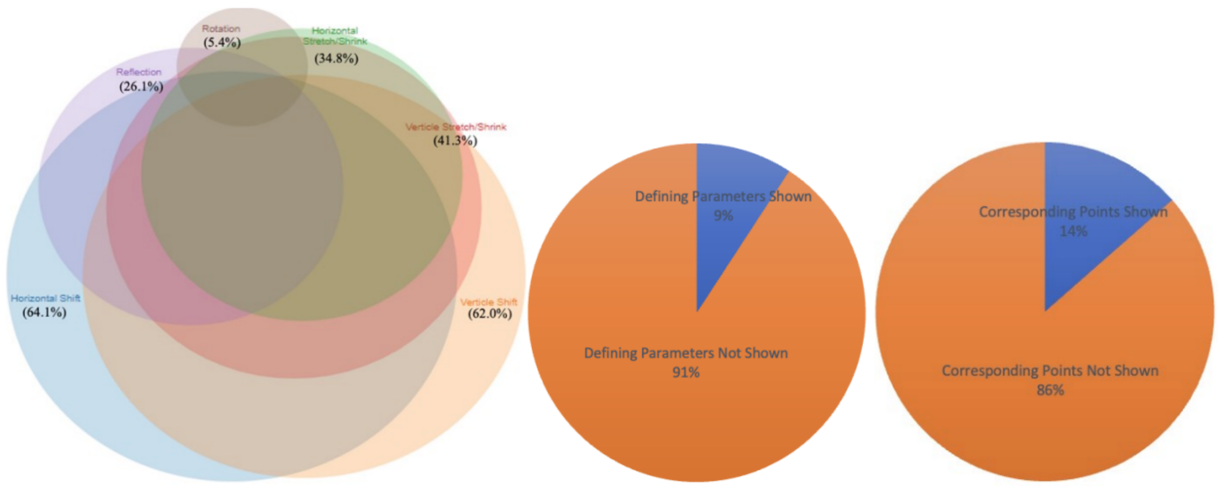
a. Function families b. Representations of parent function c. Representations of child function

Figure 1: Function Families and Their Representations in the GeoGebra Applets

Regarding the representations of a parent function, 43% of the GeoGebra applets use and only use concrete symbolic and graphical representations, 24% use only graphical representations, and 4% use and only use general symbolic and graphical representations. There are 24% of GeoGebra applets that has no representation of the parent function, which means that the parent function is not visible in these applets when the function graph is transformed. This often occurs when the GeoGebra applets only include the graph of a child function and sliders that control its coefficients. Regarding the representations of a child function, 42% of the GeoGebra applets use and only use concrete symbolic and graphical representations, 28% use and only use concrete symbolic, general symbolic, and graphical representations, and 14% use and only use general symbolic and graphical representations. A comparison of Figures 1b and 1c shows the use of only graphical representation in the parent function is more prominent than those in the child function. It also shows the use of general symbolic and graphical representations as well as the combination of concrete symbolic, general symbolic, and graphical representations is more prominent in the child function than those in the parent function.

Opportunity to Learn the Effects and Components of Transformations

73.5% of the GeoGebra applets do not specify the function transformations to be explored in the applets. These GeoGebra applets often allow their users to see the movements of a function graph by manipulating sliders but leave the users to name the specific transformations controlled by each slider. 21% of the GeoGebra clearly state the specific transformations and only allow these transformations to be explored. 5.5% of the GeoGebra applets specify particular function transformations to be explored but also allow their users to see the effects of other transformations. For instance, a GeoGebra file with $y = b \cdot f(x)$ in which b is controlled by a slider that takes values from -5 to 5 would allow its users to see the effects of reflection over the x-axis although the file might only target vertical shrink and stretch. Of the 26.5% of the GeoGebra applets that include specified function transformations, 64.1% include horizontal translation, 62.0% include vertical translation, 41.3 % include vertical stretch and shrink, 34.8 % include horizontal stretch and shrink, 26.1% include reflection over x-axis or y-axis, and 5.4 % include rotation. This indicates that more GeoGebra applets provide students with opportunities to learn function translation followed by the opportunity to learn vertical and horizontal stretch and shrink. Only a small number of GeoGebra applets provide students with opportunities to learn the effect of reflection and rotation of a function. Figure 2a also shows that most GeoGebra applets allow their users to explore more than one function transformation. This is typically done by creating multiple sliders that control the coefficients of a function and/or check boxes that show/hide different transformations.



a. Types of function transformations   b. Visibility of defining parameters   c. Visibility of corresponding points

Figure 2: Effects and Components of Transformations in the GeoGebra Applets

When analyzing the opportunity to learn components of transformations, we consider the visibility of both the defining parameters of the transformations and the corresponding points on the graphs of the parent and child functions. Only 9% of the GeoGebra applets show the defining parameters of a transformation (e.g., translation vector, line of reflection, or the center and scale of dilation) performed on the graph of function under exploration (Figure 2b). It is worth noting that among these GeoGebra files 75% of them are on function translation and therefore have the translation vector shown. Only in a few GeoGebra applets, the defining parameters of other transformations are shown. Only 14% of the GeoGebra applets show at least one pair of corresponding points on the graphs of the parent and child functions (Figure 2c).

Scaffolding to Learn Function Transformations

Only 18% of the GeoGebra applets provide scaffolding that aims to support their users to understand the targeted function transformations. This type of scaffolding varies in the amount of guidance. It can be short questions (e.g., What happens to the graph as k gets larger than 1? Is this a stretch or a shrink?) or a well-structured sequence of questions or activities that engage users in thinking about and reflecting on their interaction with the GeoGebra applet.

33% of the GeoGebra applets only describe the files and/or a direction for how users should interact with the applets. Here are some examples of this type of direction. "Use the sliders to adjust the values of a, b, c, and d. The first checkbox switches between a sine and a cosine graph. Click the other checkboxes to see the steps for graphing a function like this". "Use the sliders in the green box to adjust the values of a, w, and k. The gray graph is the graph of the parent function $y = f(x)$. The red graph is the graph of the transformed function $y = a \cdot f(x + w) + k$. To work with a different function, click the *Change Parent Function* button, and then click on the name of the function that you would like to use". This type of direction does not directly engage users in thinking about function transformations. 49% of the GeoGebra applets provide no direction or scaffolding for their users.

Discussion and Conclusion

This study aimed to examine the opportunity to learn function transformations afforded by the GeoGebra applets available on the GeoGebra website. The results show that quadratic function (often in its simplest form) is used by the largest number of GeoGebra applets to explore function transformations. Although an important concept in secondary school mathematics, quadratic function in its simplest as a parent function might not be ideal for exploring either vertical and horizontal shrink and stretch of the graph of a function or reflection of the graph of a function over the y-axis. Similarly, although it is difficult to differentiate between horizontal and vertical translations in the context of a liner function, a number of GeoGebra applets still use a linear function to explore function translation. This suggests that it is important to consider the choice of functions through which function transformations are explored. The characteristics of a function should not obscure the properties of the targeted function transformations. Moreover, the results also show that the parent and/or child functions are often provided and only a small number of GeoGebra applets allow their users to enter a function of their own choice or use a general function (Figure 1a). This raises the question of the extent to which these GeoGebra applets allow students to develop an understanding of function transformations that go beyond the function contexts in which the understanding is developed. Future design of GeoGebra applets on function transformations might consider providing users the opportunity to choose their functions to explore function transformations.

The results show that only a small number of GeoGebra applets include at least one pair of corresponding points on the graphs of the parent and child functions and clearly indicate the defining parameters of the transformations under exploration (Figures 2b and 2c). This suggests that the majority of the GeoGebra applets do not afford their users to attend to properties of transformations that act on the graph of a parent function. In fact, the majority of the GeoGebra applets focus on showing the connections between the change in the parameters of the symbolic expression a function and the movement of its graph (Figures 1b and 1c), suggesting a graphical approach to function transformations (Heid, Wilson, & Blume, 2015). Although this approach allows students to easily observe the function transformation rules in particular under the aid of technology (Göbel, 2021), it does not support students to understand why the rules work. It has been argued that mathematics action technology should serve not only as an amplifier but also as a

reorganizer (Pea, 1985; Sherman, 2014). To use technology as a reorganizer, we should consider how technology can be used to spark conceptual change in students' understanding (Roschelle, Noss, Blikstein, & Jackiw, 2017). Traditionally function transformations are presented in the context of exploring functions, in which graphical and tabular approaches are often used to introduce function transformation rules. The use of dynamic mathematical environments makes it easy to graph a function and perform transformation actions on the function graph or points on the graph. As a result, it supports the approach to function transformations proposed by Zazkis and colleagues (2003). Following this approach, GeoGebra applets can be designed to support students to develop a conceptual understanding of function transformations. These GeoGebra applets should include carefully designed scaffolding that supports students to articulate their thinking and reflect on their interaction with technology.

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